Reading for Lectures 34-35: PKT Chapter 12.

Will try for Monday?: new data sheet and draft formula sheet for final exam.

Our starting point for hydrodynamics are two equations:

Continuity equation: $\vec{\nabla}\cdot\vec{v}=0$

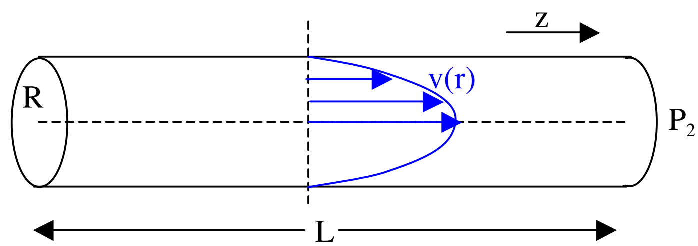
Navier-Stokes equation: $\rho\frac{\partial\vec{v}}{\partial t}+\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=-\vec{\nabla}P+\eta\nabla^2\vec{v}$ (g is set to zero, see Lect. 34)

Three examples: 1. Stokes drag (last lecture)

2. Steady-state pipe flow at arbitrary (!) Reynolds number. (full calculation)

Hagen-Poisseuille Formula: $Q=\frac{\pi}{8}\cdot\frac{R^4}{\eta}\cdot\frac{\Delta P}{L}$, where Q is volume/second thru pipe.

Geometry:
Pipe radius R, length L
End pressures P_1 (high) and P_2 (low)
(independent of r)
Program:
(a) find flow field v(r).
(b) Calculate volume flow Q through pipe.



Assume cylindrical symmetry. Steady state flow imples $\rho\frac{\partial\vec{v}}{\partial t}=0$

All velocities along the axis $\hat{z}$: $\vec{v}\left(\vec{r}\right)=v\left(r,z\right)\hat{z}$

Continuity eq: $\frac{dv}{dz}=0$, so $\vec{v}\left(\vec{r}\right)=v\left(r\right)\hat{z}$ and inertial term vanishes: $\rho\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}=\rho v\frac{\partial v}{\partial z}=0$.

We are left with $\nabla^2\vec{v}=\frac{1}{\eta}\vec{\nabla}P$.

But, $\vec{\nabla}P$ has to point along $\hat{z}$, so $\frac{\partial P}{\partial r}=0$, i.e., P is independent of r.

Thus, $\nabla^2 v=\frac{1}{\eta}\frac{dP}{dz}$, which must be independent of z, so $\frac{dP}{dz}=\frac{P_2-P_1}{L}=-\frac{\Delta P}{L}$,

where $\Delta P\equiv P_1-P_2>0$ is the (positive) pressure drop between the ends of the tube.

Finally, then $\nabla^2 v\left(r\right)=\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right)=-\frac{1}{\eta}\frac{\Delta P}{L}$.

Integrating:

$$\frac{d}{dr}\left(r\frac{dv}{dr}\right)=-\frac{r}{\eta}\frac{\Delta P}{L}$$

$$r\frac{dv}{dr}=-\frac{r^2}{2\eta}\frac{\Delta P}{L}+A$$

$$\frac{dv}{dr}=-\frac{r}{2\eta}\frac{\Delta P}{L}+\frac{A}{r}$$ (your text gets the last step f this wrong, eq. 12.21)

$$v\left(r\right)=-\frac{r^2}{4\eta}\frac{\Delta P}{L}+A\ln r+B$$

Boundary conditions are:

$\left.\frac{dv}{dr}\right|_{r=0}=0$ no singular behavior at the axis (A=0)

$v\left(R\right)=0$ no slip on walls (fixes B)

Upshot: $v(r)=\frac{(R^2-r^2)}{4\eta}\frac{\Delta P}{L}$ note parabolic velocity profile in pipe.

Finally, calculate the steady-state rate of volume flow:

$$Q=\int dA\,v=\int_0^R dr\,2\pi r\,v(r)=\frac{2\pi}{4\eta}\cdot\frac{\Delta P}{L}\int_0^R dr\,r\left(R^2-r^2\right)=\frac{\pi}{8}\cdot\frac{R^4}{\eta}\cdot\frac{\Delta P}{L}$$

This is Hagen-Poiseuille formula for laminar flow through a pipe.
Note the high power of R, which makes blood flow very sensitive to vessel dilation/contraction.
This formula is valid at high Reynolds number as well as low.
It is believed that this flow is "linearly stable" at **all** Reynolds numbers. However, in practice it is exceedingly difficult to set up the steady state at the entrance to the tube. In practice, with sufficient care, you can get laminar (i.e., turbulence-free) flow up to $R_e \sim 10^5$; however, without such care, turbulence onsets in the standard range.

3. What's special about Low-Reynolds-number flow?

- It's hard to mix. (see below) Discussion of Couette experiment.
- It's hard to swim. (see below) Reciprocal motions work for you if the "forward" stroke is significantly slower than the "backwards" stroke. The reason is that you can transfer momentum to the backwards flowing packet of water; you get the corresponding forward momentum. In viscous fluids, that backwards-flowing packet does not get away from you, so it is entirely cancelled on the reverse stroke. In high-R_e fluids, it is the inertia term which carries the packet away.
- Avoidance and drag act at long distance. This last feature is a consequence of the absence of a natural length scale beyond R in the low-Reynolds limit. Entrainment falls off as power law rather than exponential.

Discussion of "demixing" experiment: Couette Cell

At low Reynolds number

$$\rho\frac{\partial\vec{v}}{\partial t}=-\vec{\nabla}P+\eta\nabla^2\vec{v}\quad(\text{with } \nabla\cdot\vec{v}=0)$$ Equations of "creeping motion"

force term friction term

or, in steady state, $\eta\nabla^2\vec{v}=\vec{\nabla}P$.

This is invariant under $\vec{v}\leftrightarrow-\vec{v};\ P\leftrightarrow-P$ motion reversal (c.f., the inertial term!), so

(a) steady-state flow lines reverse

(b) for a symmetrical object, upstream and downstream flow lines are identical (c.f., turbulence).

Before applying this to Couette experiment, I want to develop an analog, in which the math is simple enough so you can follow it in detail:

Damped particle motion in the non-inertial limit with a time-dependent force

Consider a single particle in damped motion subject to a time-dependent force: $m\frac{dv}{dt}=-\gamma v+f(t)$.

friction term force term

Suppose f(t)=f (constant), then it is easy enough to solve.

In the long-time limit, v approaches a constant drift velocity, $v_\infty=f/\gamma$, at which the driving force and the viscous damping cancel. The full solution is $v(t)=v_\infty+(v_0-v_\infty)e^{-\frac{\gamma}{m}t}$, i.e., any initial deviation from v_∞ relaxes with a characteristic time $\tau_{relax}=\frac{m}{\gamma}$.

Now, suppose that f(t) varies with time, but slowly on the scale of τ_{relax}, so that

$$\frac{1}{f}\frac{df}{dt}\equiv\frac{1}{\tau_{driving}}<<\frac{1}{\tau_{relax}}\equiv\frac{\gamma}{m}.$$

What we expect to happen is that, after an initial transient of order τ, the motion just follows the force, according to $\boxed{v(t)=\frac{1}{\gamma}f(t)}$, so that $x(t)=x_0+\frac{1}{\gamma}\int_0^t d\bar{t}\, f(\bar{t})$. This motion is very much like the Couette cell in the sense that the particle returns to its starting point whenever $\int_0^t d\bar{t}\, f(\bar{t})=0$.

(graphical interpretation and relevance to "swimming" and to Couette)

The boxed equation is an exact solution in the strict "non-inertial" limit m=0.

However, it is not an exact solution for m>0, since $m\frac{dv}{dt}\neq 0$.

Question: Can it be a good approximate solution when m is small, except, of course, for the short initial transient?

Answer: Yes.

Suppose we eliminate the initial transient by assuming $v_0=\frac{1}{\gamma}f(0)$.

Then, what happens as the system evolves in time is that the system evolves away from the boxed solution because the mdv/dt term is not zero; however, it is continuously relaxing back towards it due to the fast exponential decay and never gets far away.

I want now to sketch a systematic expansion in which the boxed (non-inertial) equation is the lowest approximation and there is a systematic expansion about it:

$$\boxed{m\frac{dv}{dt}=f(t)-\gamma v \text{ subject to bc } f(0)=\gamma v(0).}$$ (i.e., if f(0)=0, then v(0)=0)

In this case, it's easy. We do not have to develop a perturbation theory because there is an exact solution:

It is easy to verify that the following is an exact solution of the equation plus boundary conditions:

$$v(t)=e^{-\frac{\gamma}{m}t}\left[\frac{f(0)}{\gamma}+\frac{1}{m}\int_0^t d\bar{t}f(\bar{t})e^{\frac{\gamma}{m}\bar{t}}\right].$$ (don't worry for now about where this comes from!)

Verify that, when f is time independent, this gives you back $v(t)=v_d$.

There is a nice trick for expanding this solution to see how it works.

Simply, integrate by parts under integral sign: $\begin{bmatrix} f(\bar{t}) & \frac{m}{\gamma}e^{\frac{\gamma}{m}t} \\ \frac{df}{d\bar{t}} & e^{\frac{\gamma}{m}t} \end{bmatrix}$, so

$$v(t)=e^{-\frac{\gamma}{m}t}\left[\frac{f(0)}{\gamma}+\frac{1}{m}\left\{\left(\frac{m}{\gamma}f(\bar{t})e^{\frac{\gamma}{m}\bar{t}}\right)_0^t-\frac{m}{\gamma}\int_0^t d\bar{t}\frac{df}{d\bar{t}}e^{\frac{\gamma}{m}\bar{t}}\right\}\right]=\frac{f(t)}{\gamma}-\frac{e^{-\frac{\gamma}{m}t}}{\gamma}\int_0^t d\bar{t}\frac{df}{d\bar{t}}e^{\frac{\gamma}{m}t}$$

Continuing this process:

$$\boxed{v(t)=\frac{f(t)}{\gamma}-\frac{m}{\gamma^2}\frac{df}{dt}+\frac{m}{\gamma^2}\frac{df}{dt}\bigg|_{t=0}e^{-\frac{\gamma}{m}t}}+\frac{m}{\gamma^2}e^{-\frac{\gamma}{m}t}\int_0^t d\bar{t}\frac{df}{d\bar{t}}e^{\frac{\gamma}{m}t}.$$

The ratio of second to first term is $\frac{m}{\gamma^2}\gamma\frac{1}{f}\frac{df}{dt}\sim\frac{\tau_{relax}}{\tau_{driving}}$. When this ratio is small, the expansion converges rapidly and the first term is a good approximation.

NOTE: Can also derive this expansion by a careful iteration of the original equation:

Take zeroth order approximation as $v_0(t) = \frac{1}{\gamma} f(t)$ and substitute into the equation of motion: $m\frac{dv_0}{dt} = \frac{m}{\gamma}\frac{df}{dt} \overset{?}{=} f(t) - \gamma v_0(t) = 0$.

Fails to solve the equation because of the time derivative on the left.
To make a first correction, we need to add a term to $v_0(t)$ to cancel the extra time derivative:
You first think to add a term to cancel this time derivative on the left:

$v_1(t) \overset{?}{=} \frac{1}{\gamma} f(t) - \frac{m}{\gamma^2}\frac{df}{dt}$. Check whether this is solution by substitution:

$$\frac{m}{\gamma}\frac{df}{dt} \overset{?}{=} f(t) - \gamma v_0(t) = 0$$

$$m\frac{dv_1}{dt} = \frac{m}{\gamma}\frac{df}{dt} - \frac{m^2}{\gamma^2}\frac{d^2 f}{dt^2} \overset{?}{=} f(t) - \gamma v_1(t) = 0 + \frac{m}{\gamma}\frac{df}{dt},$$

which has, indeed, canceled the first time derivative on the left.
But, there is a problem: The boundary condition at t=0 is no longer satisfied:

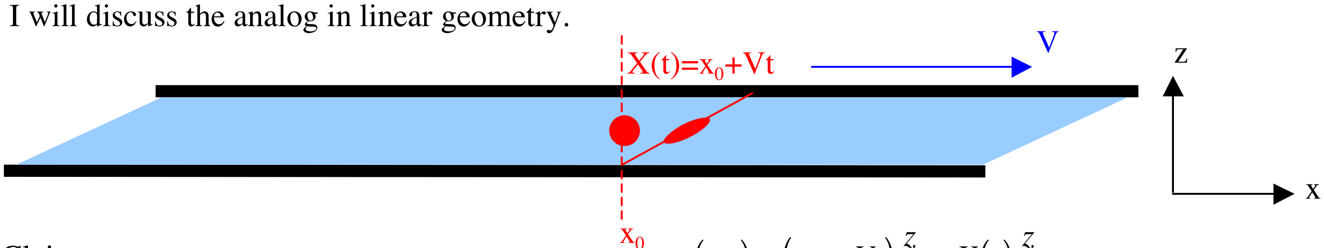
$$v_1(0) = \frac{1}{\gamma} f(0) - \frac{m}{\gamma^2}\left.\frac{df}{dt}\right|_{t=0} \neq v_0.$$

But, we can always add an additional solution of the homogeneous equation, so

$v_1(t) = \frac{1}{\gamma} f(t) - \frac{m}{\gamma^2}\frac{df}{dt} + \frac{m}{\gamma^2}\left.\frac{df}{dt}\right|_{t=0} e^{-\frac{\gamma}{m}t}$, which now obeys the bc.

(note that this is what came out of exact equation)
Can iterate this process, thus building up the full exact solution.

Now, back to the Couette cell demo done in the last lecture.
I will discuss the analog in linear geometry.



Claim: $x(z,t) = (x_0 + Vt)\frac{z}{L} = X(t)\frac{z}{L}$

When the upper-plate velocity V is constant, the simple shear flow $\vec{v} = v(z)\hat{x} = V\frac{z}{L}\hat{x}$ with P=0 satisfies the equations of steady-state creep AND, more generally, the full steady state N-S equation.
Comment: This means that an initially compact blob gets sheared into a thin sheet.
Proof:
As in pipe flow, the incompressibility condition is satisfied automatically.

Furthermore, $\eta\frac{d^2 v(z)}{dz^2} = 0$,

and it satisfies the boundary conditions at the lower and upper plates. $v(0) = 0; v(L) = V$.
Reversing the plate motion (also at constant speed) reassembles the initial blob.
But, what is less obvious is that, at low Reynolds number, ANY motion X(t) almost satisfies the creep equation (at low Reynolds number only).
Claim:
Provided accelerations are not too large, any motion which starts at X(t=0)=0 and ends at

some later time at X(T)=0 leaves the entire fluid almost in its initial state.

Proof:

Try the guess $\vec{v}(\vec{r},t) = v(z,t)\hat{x} = V(t)\frac{z}{L}\hat{x}$, where $V(t) = \frac{dX}{dt}$, i.e., $x(z,t;x_0) = x_0 + X(t)\frac{z}{L}$,

so that $x(z,t;x_0)$ returns to x_0 whenever X(t) returns to zero.

Note that this expression for v(z,t) satisfies the boundary conditions at the two plates.

But, does it satisfy the Navier-Stokes equation and incompressibility?

All the terms of the time-independent equations (including the incompressibility condition) vanish, as before; but, there is an uncancelled time derivative on the left:

$\rho\frac{\partial v}{\partial t} = \rho\frac{dV(t)}{dt}\frac{z}{L} \neq 0$. So, this is not a solution. How big is the error? Can it be fixed?

We need to solve $\rho\frac{\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}P + \eta\nabla^2\vec{v}$, which, for the shear geometry, reduces to

$$\rho\frac{\partial v}{\partial t} = \eta\frac{\partial^2 v}{\partial z^2}.$$

Thus, we can cancel the time derivative on the left by adding to the velocity field a correction:

$v(z,t) = V(t)\frac{z}{L} + \Delta v(z,t)$ with $\Delta v(z,t) = \frac{\rho L^2}{6\eta}\frac{dV(t)}{dt}\left(\frac{z}{L}\right)^3$. This extra terms leaves the incompressibility and inertial terms OK; however, it produces two new problems:

(i) Under the time derivative, it produces a new, uncompensated term of the left of

$$\rho\frac{\partial\Delta v}{\partial t} = \frac{\rho L^2}{6\eta}\frac{d^2V(t)}{dt^2}\left(\frac{z}{L}\right)^3.$$

But, this can be compensated via a new term in z^5, etc. (convergence?)

(ii) This term spoils the agreement with the upper plate boundary condition:

$$v(L,t) = V(t) + \frac{\rho L^2}{6\eta}\frac{dV(t)}{dt}.$$

But this can be compensated by correcting the coefficient of the original term, so

$$v(z,t) = \left(V(t) - \frac{\rho L^2}{6\eta}\frac{dV(t)}{dt}\right)\frac{z}{L} + \frac{\rho L^2}{6\eta}\frac{dV(t)}{dt}\left(\frac{z}{L}\right)^3.$$

You can imagine iterating to provide a fully corrected solution in this way, PROVIDED that the correction terms are small. But, are they?

Check: Suppose that V_0 is a characteristic speed of plate motion.

Then the relative size of the correction term is measured by

$\frac{1}{V_0}\cdot\frac{\rho L^2}{6\eta}\frac{dV(t)}{dt} = \frac{1}{6}\cdot\frac{\rho L V_0}{\eta}\cdot\frac{L}{V_0}\cdot\left(\frac{1}{V_0}\frac{dV}{dt}\right) \sim R_e\frac{T}{\tau}$, where T is the time for the upper plate to move a distance of the layer thickness and τ is the scale of variation of V(t).

Is this small? If so, the pure-shear solution is very good.

If we are in a very-low-Reynolds number regime, then the T/τ ratio does not have to be small.

Try realistic numbers for the Couette cell:

$$\rho \sim 10^3\ \mathrm{kg/m^3};\ L \sim \mathrm{cm} = 10^{-2}\ \mathrm{m};\ V \sim \mathrm{cm/s} = 10^{-2}\ \mathrm{m/s};\ \eta(\mathrm{glycerine}) = 1\ \mathrm{Pa\,s};\ \tau \sim 1\ \mathrm{s}$$

$$\frac{\rho L V_0}{6\eta}\cdot\frac{L}{V_0}\cdot\left(\frac{1}{V_0}\frac{dV}{dt}\right) \sim \frac{10^3\cdot 10^{-2}\cdot 10^{-2}}{6}\cdot\frac{10^{-2}}{10^{-2}}\cdot\frac{1}{1} = \frac{1}{60}$$

with some additional factors of z/L if you are not at the upper plate.

Upshot: The pure shear approximation is pretty good.